Fundamental groups of orbifolds with nef anti-canonical bundle

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Introduction

Theorem[Kobayashi 61]

Let X be a compact Kähler manifold such that $-K_X$ is ample. Then $\pi_1(X)$ is trivial.

Proof via differential geometry

- As $-K_X$ is ample, we have a closed positive $(1,1)$-form β on X such that its class is $c_1(X)$.
- Using Yau's theorem, we can find a Kähler form ω such that its Ricci form $\text{Ricci}_\omega = 2\pi\beta > 0$.
- By Myers' theorem, we know the universal covering $\tilde{X}$ of X is compact. Hence $\tilde{X} \to X$ is a finite covering.
- By comparing the Euler characteristic, we conclude the degree of $\tilde{X} \to X$ is 1.

What happens if we add singularities to X?

Theorem[Campana-Claudon 11]

Let (X, Δ) be an orbifold such that $c_1(-(K_X + \Delta)) > 0$. Then $\pi_1(X, \Delta)$ is finite.

The previous mentioned differential geometric proof can be used to prove the above theorem.

Theorem[Braun 21]

Let (X, Δ) be klt such that $-(K_X + \Delta)$ is big and nef. Then $\pi_1(X, \Delta)$ is finite.

The proof of Braun's results is very involved.
Today we will consider weakening the positive condition of $-K_X$ from ample to nef.

Theorem[Păun 1998]

Let X be a compact Kähler manifold whose anti-canonical bundle $-K_X$ is nef. Then $\pi_1(X)$ is virtually nilpotent.

We recall that a group $G = G_0$ is nilpotent if there exists an $n \in \mathbb{N}$ such that $G_n := [G_{n-1}, G]$ is trivial. A virtually nilpotent group is an extension by a finite group of a nilpotent group, *i.e.*, we have an exact sequence

$$1 \to N \to G \to F \to 1,$$

where N is nipotent and F is finite.
A fundamental result of Gromov is that for finitely generated groups, being virtually nilpotent is equivalent to having polynomial growth.

Complex Orbifolds

A complex orbifold of dimension n is a log pair (X, Δ) of dimension n such that

- Δ has standard coefficients, *i.e.*, $\Delta = \sum(1 - \frac{1}{m_i})D_i$ with $m_i \in \mathbb{N}$;
- For each $x \in X$, there exists an open $U \subset X$, an open subset $\tilde{U} \subset \mathbb{C}^n$ and a Galois analytic covering map $\phi : \tilde{U} \to U$ whose branching divisor is $B(\phi) = \Delta|_U$. The $(\tilde{U}, \phi)$ is called an orbifold chart.

Remark

- Any orbifold (X, Δ) is klt;
- If p is a general point of D_i, then it admits a chart with the form $\mathbb{C}^n \ni (z_1, ..., z_n) \to (z_1^{m_i}, z_2, ..., z_n) \in \mathbb{C}^n$. The point p is identified with 0 in the target and D_i is identified with $\{z_1 = 0\} \subset \mathbb{C}^n$.
- We have the following compatibility condition: Let $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$ be two charts such that $U \cap V \neq \emptyset$. There exists an open subset $W \subset U \cap V$, and chart $(\tilde{W}, \rho)$ and two open embeddings $\tilde{W} \to \tilde{U}$ and $\tilde{W} \to \tilde{V}$ such that the following diagram commutes

$$\begin{array}{ccccc} \tilde{U} & \longleftarrow & \tilde{W} & \longrightarrow & \tilde{V} \\ \downarrow \phi & & \downarrow \rho & & \downarrow \psi \\ U & \longleftarrow & W & \longrightarrow & V \end{array}$$

- We can define a real orbifold with underlying topological space X by giving X an atlas $\mathcal{U}$ of orbifold charts. In this case, each chart is of the form $(\tilde{U}, G, \phi : \tilde{U} \to U)$, with G a finite group acting faithfully on $\tilde{U}$ and ϕ induces an homeomorphism $\tilde{U}/G \cong U$. The charts in the atlas also need to satisfy the above compatibility condition. We often denote the orbifold $(X, \mathcal{U})$ by $\mathcal{X}$
- For an orbifold $\mathcal{X}$, we define a tensor T on $\mathcal{X}$ to be a collection of G_i-invariant tensors T_i on $(\tilde{U}_i, G_i)$ with compatibility condition. In particular, we can say orbifold Riemann metrics or orbifold Kähler forms.

Covering orbifolds

Let $\mathcal{Y}$ and $\mathcal{X}$ be two orbifolds and $p : Y \to X$ a continuous map between their underlying spaces. We say that p is a covering map if for each point $x \in X$, there exists a chart $(\tilde{U}, G, \phi)$ around x such that each component V_i of $p^{-1}(U)$ has a chart $(\tilde{U}, H_i, \psi_i)$ where H_i is a subgroup of G, $p \circ \psi_i = \phi$ and $\psi_i(\tilde{U}) = V_i$.

Theorem[Thurston 80s]

Let $\mathcal{X}$ be an orbifold. Then the universal orbifold covering $\tilde{\mathcal{X}} \to \mathcal{X}$ exists.

We will use the fact that covering maps can pull back tensors and differential forms.

Orbifold fundamental group

Let $\mathcal{X}$ be an orbifold.

- Let $p : \tilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering of $\mathcal{X}$. We define the orbifold fundamental group of $\mathcal{X}$ to be the Galois group of p and denote it by $\pi_1^{\mathrm{orb}}(\mathcal{X})$.
- If $\mathcal{X} = (X, \Delta = (1 - \frac{1}{m_j})D_j)$ is a complex orbifold, we can compute $\pi_1(X, \Delta) := \pi_1^{\mathrm{orb}}(\mathcal{X})$ by
$$\pi_1(X_{\mathrm{reg}} \setminus |\Delta|)/N,$$
where N is the normal subgroup generated by $\gamma_j^{m_j}$ with γ_j a small loop around D_j.

A good example is the weighted projective line $(\mathbb{P}^1, \frac{d-1}{d}0)$, which is a simply connected orbifold.

Metric space structure

- Let $(\mathcal{X}, g)$ be a Riemannian orbifold. We can introduce a distance d on the underlying space X.

Lemma[Bredon 72]

Let $c : [0,1] \to \mathbb{R}^n/G$ be a continuous map. If G is finite, then c admits a continuous lift to $\mathbb{R}^n$.

- Thus for any continuous curve $c : [0,1] \to X$ we can defined its length by lifting it locally. We then define the distance $d(x,y)$ to be considering the infimum of the lengths of curves connecting x and y.
- If we equip the universal covering $\tilde{\mathcal{X}}$ with the pullback metric $p^*(g)$, then $\pi_1(\mathcal{X})$ acts properly discontinuously on $\tilde{X}$ by isometries.

Metric space structure 2

- For a compact Riemannian orbifold $(\mathcal{X}, g)$, one can show that with the natural distance $\tilde{d}$ induced by $p^*(g)$, the metric space $(\tilde{X}, \tilde{d})$ is complete.
- Thus the following Gromov-Bishop theorem for orbifold holds

Theorem[Borzellino 93]

Let k be a real number. Let $(\mathcal{X}, g)$ be a complete Riemannian orbifold with $\mathrm{Ric}_g \geq (n-1)k$. Let $v(n,k,r)$ be the volume of a ball of radius r in the model space with constant curvature k. Fix a point $p \in X = |\mathcal{X}|$. The volume ratio

$$r \mapsto \frac{\mathrm{vol}_g(B(p,r))}{v(n,k,r)}$$

is a non-increasing function whose limit is $\frac{1}{|G_p|}$ as $r \to 0$.

Margulis lemma

Once we know that Gromov-Bishop theorem holds for the universal cover, we can directly applying an algebraic version of generalized Margulis lemma obtained by Breuillard-Green-Tao,
we have the following

Margulis lemma for orbifolds

Let $n \geq 1$ be an integer. There exists $\alpha = \alpha(n) > 0$ such that the following holds true. Suppose that $((X, \Delta), \omega)$ is a complete Kähler orbifold with its Ricci form bounded by $\mathrm{Ricci}_\omega \geq -(2n-1)\omega$ and Γ a subgroup of $Isom(X)$ acting properly discontinuosly by isometries on X. Then for every $x \in X$, the "*almost stabliser*"

$$\Gamma_\alpha(x) := \langle \{\gamma \in \Gamma \mid \mathrm{d}(\gamma \cdot x, x) < \alpha\} \rangle$$

is virtually nilpotent.

We first state our theorem.

Theorem[L. 22]

Let (X, Δ) be a compact Kähler orbifold such that $-(K_X + \Delta)$ is nef. Then the fundamental group $\pi_1(X, \Delta)$ is virtually nilpotent.

The idea of the proof is to find a Kähler metric ω_0 on $\mathcal{X}$, such that the for the induced metric $\tilde{d}_0$ on $\tilde{X}$, we have $\tilde{d}_0(\gamma \cdot x, x) < \alpha$ for a system of generators of $\pi_1(X, \Delta)$.

Sketch of the proof

The proof is parallel to Păun's proof in the manifold case. Except now we need to use the orbifold version of every results Păun has used.
Fix a Kähler metric ω on $\mathcal{X}$.
First we show that for $\epsilon > 0$, there exists a Kähler metric ω_ϵ cohomologous to ω, and the Ricci form of ω_ϵ satisfying

$$\mathrm{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon.$$

Sketch of the proof

By the nefness condition of $-(K_X + \Delta)$, we know for any $\epsilon > 0$, there exists an Hermitian metric h_ϵ on $K_{\mathcal{X}^{-1}}$, such that
$u_\epsilon = \Theta_{h_\epsilon} \geq -\epsilon\omega$.
It is thus sufficient to search an ω_ϵ such that

$$\text{Ricci}_{\omega_\epsilon} = -\epsilon\omega_\epsilon + \epsilon\omega + u_\epsilon. \tag{1}$$

By $\partial\bar{\partial}$-lemma $u_\epsilon = \text{Ricci}_\omega + \sqrt{-1}\partial\bar{\partial}f_\epsilon$. And to search ω_ϵ is the same as to search a potential ϕ_ϵ such that $\omega_\epsilon = \omega + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon$.
Eq.1 on ω_ϵ is thus equivalent to

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon)^n}{\omega^n} = \exp(\epsilon\phi_\epsilon - f_\epsilon). \tag{2}$$

Eq 2. has a unique solution due to Aubin-Yau's theorem for orbifold.

Sketch of the proof

- We fix a finite system of generators $\{\gamma_i\}$ of $\pi_1^{\mathrm{orb}}(\mathcal{X})$.
- Take a connected compact subset $U \subset \tilde{X}$ which contains the fundamental domain F of $\pi_1(X, \Delta)$.
- As $\{\gamma_i\}$ is finite, we may take U large enough, such that $U \cap \gamma_j U \neq \emptyset$ for all j.
- Choose a sufficiently small $\delta > 0$ such that $\delta < \frac{1}{4} \operatorname{vol}_\omega(U \cap \gamma_j U)$ and $\delta < \frac{1}{2} \operatorname{vol}_\omega F$.
- By a technical lemma due to Demailly-Peternell-Schneider, there exists a subset $U_{\epsilon,\delta} \subset U$, such that $\operatorname{diam}_{\omega_\epsilon}(U_{\epsilon,\delta}) < C_1 \delta^{\frac{1}{2}} := C$ and $\operatorname{vol}_\omega(U \setminus U_{\epsilon,\delta}) < \delta$.
- By the choice of δ, by comparing the volume, we know that $U_{\epsilon,\delta} \cap \gamma_j U_{\epsilon,\delta} \neq \emptyset$. Fix a $\tilde{x}_0 \in U_{\epsilon,\delta}$ then $d_{\omega_\epsilon}(\tilde{x}_0, \gamma_j \tilde{x}_0) < C$.

Sketch of the proof

- We set $\tilde{\omega}_\epsilon := \dfrac{\epsilon}{2n-1}\omega_\epsilon$. Then $\mathrm{Ricci}_{\tilde{\omega}_\epsilon} \geq -(2n-1)\tilde{\omega}_\epsilon$ and $\mathrm{d}_{\tilde{\omega}_\epsilon}(\tilde{x}_0, \gamma_j \tilde{x}_0) < \dfrac{\epsilon}{2n-1}C$. For ϵ sufficient small, we see that $\dfrac{\epsilon}{2n-1}C < \alpha$.

Relations with Albanese morphism

- If the Albanese morphism for a klt log pair (X, Δ) is surjective and has connected fibers, Campana showed one can compare $\pi_1(X, \Delta)$ with the Abelian group $Alb(X)$.
- In particular, when $\pi_1(X, \Delta)$ is virtually nilpotent, one can show that $\pi_1(X, \Delta)$ is virtually abelian.
- When X is projective and $-(K_X + \Delta)$ is nef, we know this holds for (X, Δ) klt. When X is compact Kähler, Păun proved its Albanese morphism is surjective. J. Cao proved that it has connected fibers outside a codimension 2 subset.

Thank you!